Set-up

- $Z_s$ is a genus 2 curve over $\mathbb{F}_q$ with $D_4 \subseteq \text{Aut}(Z_s)$:
  \[ Z_s : y^2 = r(x^2 + 1)(x^4 + sx^2 + 1), \]
  where either $r = 1$ or $r$ is a fixed nonsquare.
- This parametrizes the Moonen family M[4].
- $\text{Jac } Z_s \sim E_s^2$, where $E_s$ is the elliptic curve
  \[ E_s : y^2 = r(x + 1)(x^2 + sx + 1). \]
- Consider the degree 16 unramified cover given by the pullback

\[
\begin{array}{ccc}
Z_s^{(2)} & \longrightarrow & \text{Jac } Z_s_i \\
\downarrow & & \downarrow 2 \\
Z_{si} & \longrightarrow & \text{Jac } Z_{si}
\end{array}
\]

- We call $Z_{s_1}$ and $Z_{s_2}$ *doubly isogenous* if $\text{Jac } Z_{s_1} \sim \text{Jac } Z_{s_2}$ and $\text{Jac } Z_{s_1}^{(2)} \sim \text{Jac } Z_{s_2}^{(2)}$. 

**Lemma:** Up to isogeny, $\text{Jac } Z_{s_i}^{(2)}$ decomposes into 17 elliptic curves, including two copies of $E_s$. The 15 others are in six orbits under the $D_4$-action; five of these orbits depend on $s$.

**Question:** Describe the heuristics of doubly isogenous curves which are not Galois conjugate.
(Naive) heuristics

What one would roughly expect:

- $\#\{\text{curves } Z_s/\mathbb{F}_q\} \sim q$.
- Isogeny classes have size roughly $\sqrt{q}$.
- $\#\{\text{pairs } (Z_{s_1}, Z_{s_2})\} \sim q^2/2$.
- $\#\{\text{pairs } (Z_{s_1}, Z_{s_2}) \text{ isogenous Jacobians}\} \sim C_1 q^{3/2}$.
- Question: what is $C_1$?
- There are five additional conditions to be doubly isogenous coming from the five non-constant orbits of 2-torsion points so $\#\{\text{pairs } (Z_{s_1}, Z_{s_2}) \text{ doubly isogenous}\} \sim C_2 q^{-1}$.
- Or can we hope for more?
Data

Curves with isogenous Jacobians:

- For many primes \( p \), \( \# \{ \text{pairs } (Z_{s_1}, Z_{s_2}) \text{ isogenous} \} \sim 0.41q^{3/2} \).
- The constant \( C_1 \approx 0.41 \) is subtle to explain, since we lack information about the distribution of isogeny class sizes conditional on being in this family.

Doubly isogenous curves:

- We found more doubly isogeny curves than expected, indicating that more collisions occur than expected for isogeny classes.
- We have an explanation for this.
A family of coincidences

A one parameter family of pairs.

- We consider pairs \((Z_s, Z_{-s})\).
- In that case, three of the six orbits already match up (coincidentally).
- Two other orbits match up simultaneously.

Heuristics:

- \(#\{\text{pairs } (Z_s, Z_{-s})\} \sim q\).
- \(#\{\text{pairs } (Z_s, Z_{-s}) \text{ isogenous Jacobians}\} \sim q^{1/2}\).
- \(#\{\text{pairs } (Z_s, Z_{-s}) \text{ doubly isogenous}\} \sim q^{-1/2}\).
Discussion

Topics to investigate

- Explain the constant $C_1$?
- Could the collisions in $\mathbb{F}_q$ come from collisions in characteristic 0?
- There are additional families of collisions which appear in positive characteristic; we have found two extra genus 1 families, and there might possibly be more. We think that the majority of the collisions might come from positive characteristic families of collisions rather than characteristic 0 collisions.
- $\sim q$ collisions are found if we pull back along multiplication by $1 \pm i$ instead of multiplication by 2, this might be more fruitful.
- Do similar phenomena occur for other Moonen families?